

AAE635: Applied Microeconomic Theory

Discussion Section 7

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Midterm Review

The One-Input Maximization Problem

$$\max_x \pi(x), \text{ or } \max_x \{pg(x) - wx\}$$

- **FONC:** $pg'(x) - w = 0$ Value of the marginal product equals marginal input cost.
- **SONC:** $pg''(x) \leq 0$ or $g''(x) \leq 0$ (if $p > 0$) Additional units of x increase output, but at a non-increasing rate.
- **SOSC:** $g''(x) < 0$ Additional units of x increase output, but at a decreasing rate (diminishing marginal productivity).

Mathematical Tools

Assume that $f(x)$ is twice continuously differentiable.

- $f(x)$ is concave iff $f''(x)$ is a negative semi-definite matrix $\forall x$.
- If the function $f''(x)$ is negative definite $\forall x$, then $f(x)$ is **strictly concave**.

Homogeneity

A function $f(x)$ is *homogeneous of degree r* if, for all x and $t > 0$, $f(tx) = t^r f(x)$.

Euler's Theorem. If $f(x)$ is homogeneous of degree r and continuously differentiable, then $f'(x)x = rf(x)$, or $\sum_i (\frac{\partial f}{\partial x_i} = r(f(x))$.

- Eulers Theorem helps classify production functions based on their degree of homogeneity.

Unconstrained Optimization (N-inputs, Single-output Model)

Consider the optimization problem:

$$\max_x \{f(\mathbf{x}) : \mathbf{x} \geq 0, \mathbf{x} \in \mathbb{R}^n\}$$

Consider the first and second derivatives of $f(\mathbf{x})$:

$$f'(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right) = (1 \times n) \text{ vector of first order derivatives}$$

and

$$f''(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} = (n \times n) \text{ matrix of second order derivatives.}$$

- **FONC:** $f'(\mathbf{x}^*) = 0$ is a necessary condition for $\mathbf{x}^*$ to be an interior maximum.
- **SONC:** $f''(\mathbf{x}^*)$ must be negative semi-definite matrix.
- **SOSC:** $f''(\mathbf{x}^*)$ must be negative definite matrix.
- **Sufficient conditions:** $f'(\mathbf{x}^*) = 0$ and $f''(\mathbf{x}^*)$ negative definite matrix

Optimization under concavity

If $f(\mathbf{x})$ is concave, then $f'(\mathbf{x}) = 0$ is a necessary and sufficient condition for $\mathbf{x}^*$ to be a global interior solution.

- Some problems ask you to show that the objective function is concave (Hessian matrix negative semi-definite) before proceeding.

Profit Maximization (N-inputs, Single-output)

- Profit maximization implies technical efficiency where $y = g(x)$. If $y < g(x)$, the firm is not maximizing profit.

FONC and MR=MC

- First-order condition implies $MR = MC$.
- Also implies $MP > 0$.

Diminishing Marginal Productivity

- Equivalent to concavity of the production function: $g''(x)$ negative semi-definite.
- Given $p > 0$, $g''(x)$ negative semi-definite $\iff \pi''(x)$ negative semi-definite $\iff \pi(x)$ concave.
- Profit maximization cannot occur where $MP < 0$ or where MP increases.

Effects of Input Prices w on $\mathbf{x}^*$

$$\frac{\partial \mathbf{x}^*}{\partial w} \equiv \mathbf{x}_w^* = [\pi_{\mathbf{x}\mathbf{x}}]^{-1}$$

- Diagonal elements of a negative definite matrix are always negative, implying

$$\frac{\partial x_i^*}{\partial w_i} < 0.$$

- Input demand functions are downward sloping.
- Symmetry:

$$\frac{\partial x_i^*}{\partial w_j} = \frac{\partial x_j^*}{\partial w_i} \quad \forall i \neq j,$$

so cross-price effects may be positive or negative.

Effects of Output Price p on y^*

The effects of output price p on y^* :

$$y^*(p, w) = g(\mathbf{x}^*(p, w))$$

- $\frac{\partial y^*}{\partial p} = \frac{\partial g}{\partial \mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial p} \geq 0$.
- Intuition: Increasing price stimulates production.

Homogeneity Property

- $\mathbf{x}^*(p, w)$ is homogeneous of degree 0 in prices (p, w) .

- Intuition: profit-maximizing behavior is invariant to proportional changes in all prices.
- By Eulers theorem, the sum of price elasticities across all prices equals 0.
- Similarly, $y^*(p, w)$ is homogeneous of degree 0 in prices, and the sum of its price elasticities is 0.

The LeChatelier Principle

- **Unconditional (Long-run):** All inputs fully adjustable.
- **Conditional (Short-run):** Some inputs fixed.
- **Implications:**
 - Unconditional price effects are larger than conditional ones.
 - Long-run decisions are more price-responsive than short-run ones.
 - Regulatory or technical constraints reduce the ability of firms to respond to price changes.

The Lagrangean Approach

We want to maximize or minimize $f(\mathbf{x})$ subject to some constraint $h(\mathbf{x}) = 0$.

$$\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda h(\mathbf{x})$$

By rearranging the constraint function h , one can express

$$x_a = s(x_b)$$

That is, x_a is written as a function of the remaining variables x_b .

Second order necessary condition (SONC)

The SONC (5) can be alternatively written as

$$\begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*)^T & \mathbf{I}_{n-m} \end{bmatrix} \mathbf{L}_{xx}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix} = \text{symmetric, negative semi-definite matrix,} \quad (10)$$

where

$$\mathbf{s}_b = \frac{\partial \mathbf{s}}{\partial \mathbf{x}_b}$$

is a $m \times (n - m)$ matrix, $\mathbf{I}_{n-m}$ is an identity matrix of dimension $(n - m)$, and

$$\mathbf{L}_{xx} \equiv \frac{\partial^2 \mathcal{L}}{\partial \mathbf{x}^2} = \mathbf{f}_{xx} + \boldsymbol{\lambda}^T \cdot \mathbf{h}_{xx}$$

is an $(n \times n)$ matrix.

Cost Minimization

A firm produces a single output y using multiple inputs $\mathbf{x}$:

$$\min_{\mathbf{x}} w_1 x_1 + w_2 x_2 + \cdots + w_n x_n \quad \text{s.t.} \quad f(\mathbf{x}) = y$$

Constraint Qualification (CQ) Regularity conditions ensure that the constraint is differentiable and the gradients are linearly independent.

Cost-minimizing behavior rules out production regions where any inputs marginal product is negative.

Lagrangian Function

$$\mathcal{L} = w_1 x_1 + \cdots + w_n x_n - \lambda [f(\mathbf{x}) - y]$$

First-Order Necessary Conditions (FONC)

$$\frac{\partial \mathcal{L}}{\partial x_i} = w_i - \lambda \frac{\partial f}{\partial x_i} = 0, \quad i = 1, \dots, n$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = f(\mathbf{x}) - y = 0$$

From FONC we can derive the marginal rate of technical substitution:

$$\text{MRTS}_{ij} = \frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}} = \frac{w_i}{w_j}$$

which represents the rate at which one input can be substituted for another while keeping output constant.

Duality

We start from an indirect objective function, $V(\alpha)$.

$\Rightarrow$ Analyze properties of $x^*(\alpha)$ and $V(\alpha)$

1. **Envelope Theorem:** The derivative of an indirect objective function with respect to

parameters equals the partial derivative of a Lagrangian function:

$$\frac{dV(\alpha)}{d\alpha} = \frac{\partial L}{\partial \alpha}$$

This is used to obtain the three results below.

2. Interpretation of (λ) , the Lagrange multiplier:

Suppose that

- the objective function f does not depend on a parameter α , and
- the constraint function is $h(x, \alpha) = \alpha - g(x)$

Then, the Lagrangian multiplier is the derivative of an indirect objective function with respect to α : $\frac{dV(\alpha)}{d\alpha} = \lambda^*$.

How is it important? Remember the cost minimization problem in which the objective function is $f(x) = w \cdot x$ and the constraint function is $h(x, \bar{y}) = \bar{y} - g(x)$.

Then, the interpretation of the Lagrangian multiplier is that “the increase in total cost in response to a marginal increase in the target production”.

3. Hotelling’s Lemma: A partial derivative of an indirect profit function with respect to the price of netput j is a *netput function* of j :

$$\frac{\partial \pi(p)}{\partial p_j} = y_j(p).$$

So, if we take a partial derivative of an indirect profit function with respect to an *output price*, then we get an **output function**.

If we take a partial derivative of an indirect profit function with respect to an *input price*, then we get the **negative of an input function**.

4. Shephard’s Lemma: A partial derivative of an indirect cost function with respect to the price of input j is an *input demand function* of j :

$$\frac{\partial C(w, y)}{\partial w_j} = x_j^c(w, y).$$

So, if we have an indirect cost function, we can take a derivative with respect to an input price and obtain an input function.

Cost Minimization vs. Profit Maximization

For a competitive firm:

$$\pi(p, w) = p y^*(p, w) - C(w, y^*(p, w))$$

The same duality relationships hold for a monopoly as well.

Returns to Scale

A production technology exhibits:

$$f(t\mathbf{x}) = t^r f(\mathbf{x})$$

That is, the technology shows

- Increasing returns to scale (IRTS) if $r > 1$,
- Constant returns to scale (CRTS) if $r = 1$,
- Decreasing returns to scale (DRTS) if $r < 1$.

Scale Elasticity Differentiating both sides with respect to t and evaluating at $t = 1$:

$$\sum_i \frac{\partial f}{\partial x_i} x_i = r f(\mathbf{x})$$

Thus, scale elasticity $SE = r$ measures the proportional change in output due to a proportional change in all inputs.

Input Substitution

The strength of the substitution between inputs i and j can be measured by the Allen elasticity of substitution.

Definition: The *Allen elasticity of substitution* (AES) between inputs i and j is defined as

$$\sigma_{ij} = \frac{\partial x_i^c}{\partial w_j} \cdot \frac{C}{x_i^c x_j^c}, \quad i \neq j; \quad i, j = 1, \dots, n.$$

Practice Question

Consider a firm with the production function

$$y = f(x_1, x_2) = 4x_1^{\frac{1}{4}}x_2^{\frac{1}{4}}.$$

The markets for the output and the inputs are competitive, and the prices are p and $\mathbf{w} = (w_1, w_2)$.

1. What kind of returns to scale does this production technology exhibit?
Does this production technology exhibit increasing, constant, or decreasing returns to scale?
2. Set up a Lagrangian function and find the firm's conditional input demand functions $x_i^c(w, y)$ ($i = 1, 2$) and the Lagrangian multiplier λ^c .
3. Derive the indirect cost function $C(w, y)$.
4. Verify that the Shephard's lemma holds for both inputs.
5. Verify that the partial derivative of the indirect cost function with respect to y equals the Lagrangian multiplier:

$$\frac{\partial C(w, y)}{\partial y} = \lambda^c.$$